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$\mathbf{C}^2$-SADDLE METHOD AND BEUKERS' INTEGRAL

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ABSTRACT. We give good non-quadraticity measures for the values of logarithm at specific rational points by modifying Beukers' double integral. The two-dimensional version of the saddle method, which we call $\mathbf{C}^2$-saddle method, is applied.

0. INTRODUCTION

F. Beukers [1] has introduced the following double and triple integrals:

$$\iint_S \frac{L(x)(1-y)^n}{1-xy}\,dx\,dy \quad \text{and} \quad \iiint_B \frac{L(x)L(y)}{1-u(1-xy)}\,du\,dx\,dy, \tag{0.1}$$

giving elegant short proofs of the irrationality of $\zeta(2) = \pi^2/6$ and $\zeta(3)$ respectively, where $L(x) = (x^n(1-x)^n)^{(n)}/n!$ is the Legendre polynomial on the unit interval, $S = [0,1]^2$ and $B = [0,1]^3$. These integrals are very important in the arithmetical study of $\zeta(2)$ and $\zeta(3)$, since there exist certain modifications of the integrands in (0.1), which produce fairly good irrationality measures for them. (See G. Rhin and C. Viola [10] and the author [7] for $\zeta(2)$, the author [4] for $\zeta(3)$.)

The aim of this paper is to show that another modification of the double integral in (0.1) can produce good non-quadraticity measures for the values of logarithm at specific rational points. For example, it will be shown that there exists an effective constant H_0 satisfying

$$\left|\log 2 - \xi\right| \geq H^{-25.0463}$$

for any quadratic number ξ with $H \equiv H(\xi) \geq H_0$, where $H(\xi)$ is the usual height of ξ, the maximum of absolute values of the coefficients of its minimal polynomial. (Shortly we say that $\log 2$ has a non-quadraticity measure 25.0463.)

Concerning non-quadraticity measures of $\log 2$, H. Cohen [2] obtained the measure 287.819 by using some linear recurrence. Later E. Reyssat [9] obtained the measure 105 by considering the classical Padé approximation formula to logarithms. Our result mentioned above hence improves the earlier measures.

Let us explain briefly how we modify Beukers' integral. For any positive integer k, let R_k be the rectangular region $(1, (k+1)/k) \times (k/(k+1), 1)$. We then consider

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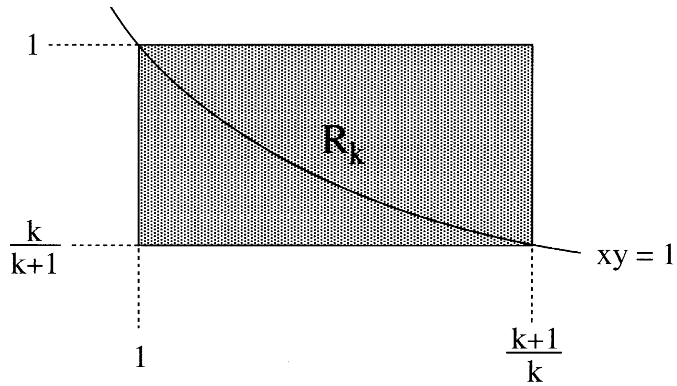

FIGURE 1.

the double integral

$$J(z) = \iint\limits_{R_k} \frac{P(x)Q(y)}{1 - xyz}\, dx\, dy,$$

where $P(x)$ and $y^n Q(y)$ are some polynomials with integral coefficients for some $n \in \mathbf{N}$. Of course, the value $J(1)$ does not exist, since the hyperbola $xy = 1$ crosses the inside of R_k (Figure 1). However it can be seen that $J(z)$ is analytic in the complex plane $\mathbf{C}$ with a branch cut $[k/(k+1), \infty)$ and that $J(z)$ has the limit I as z tends to 1 along a continuous curve lying in the upper-half plane. By virtue of Cauchy's theorem this limit may be expressed as a double curvilinear integral

$$I = \iint\limits_{\alpha \times \beta} \frac{P(w)Q(z)}{1 - wz}\, dw\, dz \tag{0.2}$$

for suitable paths $\alpha,\ \beta$ with the initial points $1,\ k/(k+1)$ and the terminal points $(k+1)/k, 1$ respectively. In consequence not only the integrand but also the integral region of Beukers' integral are modified.

Expanding the product $P(w)Q(z)$ in (0.2), we will get

$$I = a\left\{\frac{1}{2}\log^2\!\left(1 + \frac{1}{k}\right) - \pi i \log\left(1 + \frac{1}{k}\right)\right\} + b\left\{\log\left(1 + \frac{1}{k}\right) - \pi i\right\} + c$$

for some (real) rationals a, b and c. The notation $\log^n x$ is used to mean $(\log x)^n$. Taking the real and imaginary parts of the above equality we then have

$$\begin{cases} a\log\left(1 + \dfrac{1}{k}\right) + b = -\dfrac{1}{\pi}\,\mathrm{Im}(I), \\ a\log^2\!\left(1 + \dfrac{1}{k}\right) - 2c = -2\,\mathrm{Re}(I) - \dfrac{2}{\pi}\log\left(1 + \dfrac{1}{k}\right)\mathrm{Im}(I), \end{cases} \tag{0.3}$$

a simultaneous rational approximation to $\log(1 + 1/k)$ and $\log^2(1 + 1/k)$.

The remainder terms in (0.3) are essentially dominated by $|I|$. In estimating $|I|$ and the common coefficient a in (0.3) we need a two-dimensional version of the usual saddle method, which we call the $\mathbf{C}^2$-saddle method. It seems to be difficult to obtain even a good upper estimate of $|I|$ without using the $\mathbf{C}^2$-saddle method. In the next section we investigate this method in detail, which may possess a variety of applications in analysis in general. The usage of the steepest linear paths will make the argument simpler, even in the one-dimensional case described in Dieudonné's book [3]. To establish the "$\mathbf{C}^n$-saddle method" may be an interesting problem itself.

1. The $\mathbf{C}^2$-saddle method

We first give several definitions and notations. Let $f(z)$ be a non-constant analytic function on an open region $D \subset \mathbf{C}$. A point $z_0 \in D$ is said to be a *saddle* of $f(z)$ provided that $f(z_0) \neq 0$ and $f'(z_0) = 0$. (This terminology will be justified by the fact that the point $(z_0, |f(z_0)|) \in \mathbf{C} \times \mathbf{R}$ is actually a saddle of the surface defined by $|f(z)|$.) The *order* of a saddle z_0 is the smallest integer $m \geq 2$ satisfying $f^{(m)}(z_0) \neq 0$, which is denoted by $\mathrm{ord}(f; z_0)$. A saddle z_0 with $\mathrm{ord}(f; z_0) = 2$ is said to be *normal*. The *valley set* $V(f; z_0)$ of a saddle z_0 of $f(z)$ is the union of all connected components K of $\{z \in D; |f(z)| < |f(z_0)|\}$ satisfying $z_0 \in \partial K$. Obviously the number of such connected components is at most $\mathrm{ord}(f; z_0)$. A path means a continuous mapping into $\mathbf{C}$ defined on an interval of $\mathbf{R}$ which is assumed to be piecewise-continuously differentiable. The path $\sigma : [-s_0, s_0] \to \mathbf{C}$ defined by

$$\sigma(s) = \begin{cases} z_0 + e^{i\theta_+} s & (0 \leq s \leq s_0), \\ z_0 - e^{i\theta_-} s & (-s_0 \leq s < 0), \end{cases}$$

for some $s_0 > 0$, is said to be the *steepest piecewise-linear path* through z_0 provided that $\sigma([-s_0, s_0] \backslash \{0\}) \subset V(f; z_0)$, $\theta_+ \not\equiv \theta_- \pmod{2\pi}$ and $\omega + m\theta_\pm \equiv 0 \pmod{2\pi}$ where $m = \mathrm{ord}(f; z_0)$ and $\omega = \arg(-f^{(m)}(z_0)/f(z_0))$. The steepest piecewise-linear path σ satisfying $\theta_+ - \theta_- \equiv \pi \pmod{2\pi}$, which is simply written as $\sigma(s) = z_0 + e^{i\theta_+} s$ for $-s_0 \leq s \leq s_0$, is called the *steepest linear path* through z_0. If $\mathrm{ord}(f; z_0)$ is odd, then there is no steepest linear path through z_0. Note that if z_0 is normal, then the steepest piecewise-linear path through z_0 becomes necessarily linear.

We now consider the following integral:

$$I_n = \iint_{\alpha \times \beta} g(w, z)\big(f(w, z)\big)^n \, dw \, dz, \tag{1.1}$$

where $f(w, z), g(w, z)$ are analytic functions in w and z on an open region $\Delta \subset \mathbf{C} \times \mathbf{C}$ and $\alpha, \beta : (-1, 1) \to \mathbf{C}$ are paths with $\alpha \times \beta \subset \Delta$. We assume that (1.1) converges absolutely as a double curvilinear integral; hence by Fubini's theorem we have

$$I_n = \int_\alpha \left(\int_\beta g(w, z)\big(f(w, z)\big)^n dz \right) dw = \int_\beta \left(\int_\alpha g(w, z)\big(f(w, z)\big)^n dw \right) dz. \tag{1.2}$$

For the asymptotic study of I_n as n tends to infinity we need several assumptions. We first impose the following

Hypothesis A. *There exists an analytic function $w(z)$ on an open region $D \subset \mathbf{C}$ satisfying $(w(z), z) \in \Delta$, $f(w(z), z) \neq 0$ and $\frac{\partial f}{\partial w}(w(z), z) = 0$ for any $z \in D$.*

If $f(w, z)$ is a rational function in w and z, then $w(z)$ is a root of some algebraic equation on w with polynomial coefficients in z. Differentiating the last equality we get

$$w'(z) \frac{\partial^2 f}{\partial w^2}(w(z), z) + \frac{\partial^2 f}{\partial w \partial z}(w(z), z) \equiv 0. \tag{1.3}$$

Let $\mathrm{Hess}_f(w,z) = \frac{\partial^2 f}{\partial w^2}(w,z)\frac{\partial^2 f}{\partial z^2}(w,z) - \left(\frac{\partial^2 f}{\partial w \partial z}(w,z)\right)^2$ be the Hessian of $f(w,z)$. We next impose

Hypothesis B. There exists a point $z_0 \in D$ such that

$$\frac{\partial f}{\partial z}(w_0, z_0) = 0, \quad \frac{\partial^2 f}{\partial w^2}(w_0, z_0) \neq 0 \quad \text{and} \quad \mathrm{Hess}_f(w_0, z_0) \neq 0,$$

where $w_0 = w(z_0)$.

Putting $F(z) = f(w(z), z)$, we get $F(z) \neq 0$, $F'(z) = \frac{\partial f}{\partial z}(w(z), z)$ and

$$F''(z) = w'(z)\frac{\partial^2 f}{\partial w \partial z}(w(z), z) + \frac{\partial^2 f}{\partial z^2}(w(z), z)$$

for any $z \in D$; it hence follows from (1.3) and Hypothesis B that z_0 is a normal saddle of $F(z)$. We also impose

Hypothesis C. There exists a path $\gamma : (-1, 1) \to D$ with $\alpha \times \gamma \subset \Delta$ satisfying $\gamma(0) = z_0$, $\gamma((-1,1)\backslash\{0\}) \subset V(F; z_0)$ and

$$\int_\beta g(w,z)\big(f(w,z)\big)^n dz = \int_\gamma g(w,z)\big(f(w,z)\big)^n dz \tag{1.4}$$

for an arbitrarily fixed $w \in \alpha$.

Let $N_\varepsilon(z)$ be an open disk centered at z with radius ε. By modifying the path γ slightly in $N_\varepsilon(z_0) \cap V(F; z_0)$ for a sufficiently small $\varepsilon > 0$, we can assume, without loss of generality, that the restriction of γ to $[-t_0, t_0]$ is the steepest linear path through z_0 for some $t_0 > 0$. That is, $\gamma(t) = z_0 + e^{i\theta_0}t$ for $|t| \leq t_0$ with

$$\theta_0 = \epsilon\pi - \frac{1}{2}\arg\left(-\frac{\mathrm{Hess}_f(w_0, z_0)}{f(w_0, z_0)\frac{\partial^2 f}{\partial w^2}(w_0, z_0)}\right)$$

where $\epsilon \in \{0, 1\}$ depends on the direction of the path γ. Note that (1.4) is not a direct consequence of Cauchy's theorem since the integral is improper. Usually one must pass to the limit from paths in the usual sense to which Cauchy's theorem can be applied. We hence get from (1.2) and (1.4)

$$I_n = \int_\alpha \left(\int_\gamma g(w,z)\big(f(w,z)\big)^n dz\right) dw = \int_\gamma \left(\int_\alpha g(w,z)\big(f(w,z)\big)^n dw\right) dz \tag{1.5}$$

by assuming the absolute convergence as a double curvilinear integral over $\alpha\times\gamma$. For an arbitrarily fixed $z \in \gamma$, we next consider the analytic function $f_z(w) \equiv f(w,z)$ in the open region $D_z = \{w \in \mathbf{C}; (w,z) \in \Delta\}$ by imposing the following

Hypothesis D. $f_z(w)$ is not constant for any $z \in \gamma$.

Then the point $w(z)$ is a saddle of $f_z(w)$ for any $z \in \gamma$. Moreover there exists a $t_1 \in (0, t_0]$ such that $w(z)$ is a normal saddle of $f_z(w)$ for any $z = \gamma(t)$ with $|t| \leq t_1$, since $f''_{z_0}(w_0) = \frac{\partial^2 f}{\partial w^2}(w_0, z_0) \neq 0$ by Hypothesis B. We further impose

Hypothesis E. For an arbitrarily fixed $z \in \gamma$, there exists a path $\delta_z : (-1, 1) \to D_z$ satisfying $\delta_z(0) = w(z)$, $\delta_z((-1,1)\backslash\{0\}) \subset V(f_z; w(z))$ and

$$\int_\alpha g(w,z)\big(f(w,z)\big)^n dw = \int_{\delta_z} g(w,z)\big(f(w,z)\big)^n dw. \tag{1.6}$$

Furthermore $\delta_z'(0)$ is continuous for $z = \gamma(t)$ in a small neighborhood of $t = 0$.

Thus there exists a sufficiently small $t_2 \in (0, t_1]$ such that, for any $z = \gamma(t)$ with $|t| \le t_2$, the path δ_z can be modified slightly in $N_{\varepsilon(z)}(w(z)) \cap V(f_z; w(z))$ for some $\varepsilon(z) > 0$ so that the restriction of δ_z to $[-s(z), s(z)]$ is the steepest linear path through $w(z)$ for some $s(z) > 0$ and that $\delta_z'(0)$ is continuous in $t \in [-t_2, t_2]$. That is, we can put $\delta_{\gamma(t)}(s) = w(\gamma(t)) + e^{i\varphi(t)}s$ for any $|s| \le s(\gamma(t))$ and $|t| \le t_2$, with

$$\varphi(t) = \epsilon'\pi - \frac{1}{2}\arg\left(-\frac{\frac{\partial^2 f}{\partial w^2}(w(\gamma(t)), \gamma(t))}{f(w(\gamma(t)), \gamma(t))}\right),$$

where $\epsilon' \in \{0, 1\}$ depends only on the direction of the path δ_{z_0}, not depending on t by the continuity of $\delta_z'(0)$. Moreover, since $(w_0, z_0) \in \Delta$, there exist some $t_3 \in (0, t_2]$ and a positive constant ρ such that $\overline{N_{2\rho}\big(w(\gamma(t))\big)} \subset D_{\gamma(t)}$ for any $|t| \le t_3$. Let $M_\rho(t)$ be the maximum of $|f_{\gamma(t)}(w)|$ on $\overline{N_\rho\big(w(\gamma(t))\big)}$. Clearly $M_\rho(t)$ is continuous in $t \in [-t_3, t_3]$. Then we have

$$\begin{aligned}&\Big|f_{\gamma(t)}\big(w(\gamma(t)) + e^{i\varphi(t)}s\big)\Big| \\ &\qquad \le \big|f_{\gamma(t)}\big(w(\gamma(t))\big)\big| - \frac{1}{2}\big|f''_{\gamma(t)}\big(w(\gamma(t))\big)\big|\, s^2 + M_\rho(t)\sum_{\ell=3}^{\infty}\left(\frac{|s|}{\rho}\right)^{\ell};\end{aligned}$$

hence $w(\gamma(t)) + e^{i\varphi(t)}s \in V\big(f_{\gamma(t)}; w(\gamma(t))\big)$ whenever

$$0 < |s| < \frac{\rho^3\big|f''_{\gamma(t)}\big(w(\gamma(t))\big)\big|}{2M_\rho(t) + \rho^2\big|f''_{\gamma(t)}\big(w(\gamma(t))\big)\big|}.$$

Since the right-hand side has a positive minimum as $t \in [-t_3, t_3]$ varies, we can assume that $s(\gamma(t)) \ge s_0$ for any $|t| \le t_3$ with a sufficiently small $s_0 > 0$. We have from (1.5) and (1.6)

$$I_n = \int_\gamma \left(\int_{\delta_z} g(w, z)\big(f(w, z)\big)^n dw\right)dz.$$

Finally we impose the following

Hypothesis F. $g(w_0, z_0) \ne 0$ and $\int_\gamma\big(\int_{\delta_z} |g(w, z)||f(w, z)|^\ell |dw|\big)|dz| < \infty$ for some $\ell \in \mathbf{N}_0$.

Under all the assumptions mentioned above we are now in position to determine the principal part of I_n as n tends to infinity. Putting $\varphi_0 = \varphi(0)$, it is easily seen that

$$\begin{aligned}\delta_{\gamma(t)}(s) &= w(z_0 + e^{i\theta_0}t) + e^{i\varphi(t)}s \\ &= w_0 - \frac{\frac{\partial^2 f}{\partial w \partial z}(w_0, z_0)}{\frac{\partial^2 f}{\partial w^2}(w_0, z_0)}e^{i\theta_0}t + e^{i\varphi_0}s + O(s^2 + t^2)\end{aligned}$$

as $s^2+t^2 \to 0$. Hence it follows from Taylor's formula that

$$\begin{aligned} f(\delta_{\gamma(t)}(s),\gamma(t)) &= f(w_0,z_0) + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}(w_0,z_0)\left(\delta_{\gamma(t)}(s)-w_0\right)^2 \\ &\quad + \frac{\partial^2 f}{\partial w \partial z}(w_0,z_0)\left(\delta_{\gamma(t)}(s)-w_0\right)(\gamma(t)-z_0) \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial z^2}(w_0,z_0)(\gamma(t)-z_0)^2 + O\left(|s|^3+|t|^3\right) \\ &= f(w_0,z_0)\left(1-A_0 s^2 - B_0 st - C_0 t^2\right) + O\left(|s|^3+|t|^3\right), \end{aligned}$$

where

$$A_0 = -e^{2i\varphi_0}\frac{\frac{\partial^2 f}{\partial w^2}(w_0,z_0)}{2f(w_0,z_0)} = \frac{1}{2}\left|\frac{\frac{\partial^2 f}{\partial w^2}(w_0,z_0)}{f(w_0,z_0)}\right| > 0,$$

$$B_0 = e^{i(\theta_0+\varphi_0)}\frac{\frac{\partial^2 f}{\partial w^2}(w_0,z_0)}{2f(w_0,z_0)}\cdot\frac{2\frac{\partial^2 f}{\partial w\partial z}(w_0,z_0)}{\frac{\partial^2 f}{\partial w^2}(w_0,z_0)} - e^{i(\varphi_0+\theta_0)}\frac{\frac{\partial^2 f}{\partial w\partial z}(w_0,z_0)}{f(w_0,z_0)} = 0$$

and

$$C_0 = -e^{2i\theta_0}\frac{\mathrm{Hess}_f(w_0,z_0)}{2f(w_0,z_0)\frac{\partial^2 f}{\partial w^2}(w_0,z_0)} = \frac{1}{2}\left|\frac{\mathrm{Hess}_f(w_0,z_0)}{f(w_0,z_0)\frac{\partial^2 f}{\partial w^2}(w_0,z_0)}\right| > 0.$$

Therefore there exist some $s_1 \in (0,s_0]$, $t_4 \in (0,t_3]$ and $\lambda_0 > 0$ satisfying

$$|f(\delta_{\gamma(t)}(s),\gamma(t))| \le |f(w_0,z_0)|\left(1-\lambda_0 s^2 - \lambda_0 t^2\right) \tag{1.7}$$

for any $(s,t) \in [-s_1,s_1]\times[-t_4,t_4] \equiv \Omega_0$.

Now it is easily seen from Hypotheses C and E on the paths that

$$|f(\delta_{\gamma(t)}(s),\gamma(t))| \le |f(\delta_{\gamma(t)}(0),\gamma(t))| = |f(w(\gamma(t)),\gamma(t))| \le |f(w_0,z_0)|$$

for any $-1 < s,t < 1$ and that $|f(\delta_{\gamma(t)}(s),\gamma(t))| = |f(w_0,z_0)|$ if and only if $(s,t) = (0,0)$. This implies that there exists some $\mu_0 \in (0,1)$ satisfying

$$|f(\delta_{\gamma(t)}(s),\gamma(t))| \le \mu_0 |f(w_0,z_0)| \tag{1.8}$$

for any $(s,t) \in (-1,1)^2\backslash\Omega_0$. Let $n_0 \ge 3$ be the least integer satisfying $(\log n_0)/\sqrt{n_0} \le \min\{s_1, t_4, \sqrt{(1-\mu_0)/\lambda_0}\}$ and put $\Omega_n = [-(\log n)/\sqrt{n}, (\log n)/\sqrt{n}\ ]^2$ for any $n \ge n_0$. Since $s^2+t^2 > (\log^2 n)/n$ for $(s,t) \notin \Omega_n$, we get from (1.7) and (1.8)

$$|f(\delta_{\gamma(t)}(s),\gamma(t))| \le \left(1-\lambda_0\frac{\log^2 n}{n}\right)|f(w_0,z_0)|$$

for any $(s,t) \in (-1,1)^2\backslash\Omega_n$. Therefore, taking account of Hypothesis F and the following estimates

$$\left(1-\lambda_0\frac{\log^2 n}{n}\right)^n = O\left(e^{-\lambda_0 \log^2 n}\right),$$

$$g(\delta_{\gamma(t)}(s),\gamma(t))\delta'_{\gamma(t)}(s)\gamma'(t) = e^{i(\theta_0+\varphi_0)}g(w_0,z_0) + O\left(\frac{\log n}{\sqrt{n}}\right),$$

$$\left(1+\frac{O\left(|s|^3+|t|^3\right)}{1-A_0s^2-C_0t^2}\right)^n = 1 + O\left(\frac{\log^3 n}{\sqrt{n}}\right)$$

for any $(s,t) \in \Omega_n$, we have

$$\begin{aligned} I_n &= \iint_{\Omega_n} g(\delta_{\gamma(t)}(s), \gamma(t))\big(f(\delta_{\gamma(t)}(s), \gamma(t))\big)^n \delta'_{\gamma(t)}(s)\gamma'(t)\ ds\, dt \\ &\quad + O\left(e^{-\lambda_0 \log^2 n}|f(w_0,z_0)|^n\right) \\ &= e^{i(\theta_0+\varphi_0)} g(w_0,z_0)\big(f(w_0,z_0)\big)^n \iint_{\Omega_n} \left(1 - A_0 s^2 - C_0 t^2\right)^n ds\, dt \left(1 + O\left(\frac{\log^3 n}{\sqrt{n}}\right)\right) \end{aligned}$$

as $n \to \infty$, since $O\left(e^{-\lambda_0 \log^2 n}\right)$ is negligible compared to $O\left((\log^3 n)/\sqrt{n}\right)$. Finally, substituting $\sqrt{A_0 n}\, s = u$ and $\sqrt{C_0 n}\, t = v$, we get

$$\begin{aligned} \iint_{\Omega_n} \left(1 - A_0 s^2 - C_0 t^2\right)^n ds\, dt &= \frac{1}{\sqrt{A_0 C_0}\, n} \iint_{\Omega_n^*} \left(1 - \frac{u^2+v^2}{n}\right)^n du\, dv \\ &= \frac{1}{\sqrt{A_0 C_0}\, n} \iint_{\Omega_n^*} e^{-u^2-v^2}\left(1 + O\left(\frac{\log^4 n}{n}\right)\right) du\, dv \\ &= \frac{1}{\sqrt{A_0 C_0}\, n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2-v^2}\, du\, dv \left(1 + O\left(\frac{\log^4 n}{n}\right)\right) \\ &= \frac{\pi}{\sqrt{A_0 C_0}\, n}\left(1 + O\left(\frac{\log^4 n}{n}\right)\right) \end{aligned}$$

as $n \to \infty$, where $\Omega_n^* = [-\sqrt{A_0}\log n, \sqrt{A_0}\log n] \times [-\sqrt{C_0}\log n, \sqrt{C_0}\log n]$.

We thus conclude that

$$I_n = d_0 \frac{(f(w_0,z_0))^n}{n}\left(1 + O\left(\frac{\log^3 n}{\sqrt{n}}\right)\right)$$

where

$$\begin{aligned} d_0 &= e^{i(\theta_0+\varphi_0)}\, g(w_0,z_0) \frac{\pi}{\sqrt{A_0 C_0}} \\ &= e^{i(\theta_0+\varphi_0)}\, g(w_0,z_0) \frac{2\pi |f(w_0,z_0)|}{\sqrt{|\operatorname{Hess}_f(w_0,z_0)|}}. \end{aligned}$$

In particular, we get

$$|I_n| = 2\pi \frac{|g(w_0,z_0)|}{\sqrt{|\operatorname{Hess}_f(w_0,z_0)|}} \cdot \frac{|f(w_0,z_0)|^{n+1}}{n}\left(1 + O\left(\frac{\log^3 n}{\sqrt{n}}\right)\right) \tag{1.9}$$

and

$$\arg(I_n) \equiv \theta_0 + \varphi_0 + \arg(g(w_0,z_0)) + n \arg(f(w_0,z_0)) + O\left(\frac{\log^3 n}{\sqrt{n}}\right) \pmod{2\pi}$$

as $n \to \infty$. Note that all the constants in O-symbols in the above estimates can be effectively computable.

2. Arithmetical lemmas

Put $\zeta = (k+1)/k$ for an arbitrarily fixed positive integer k. To calculate the limit of $J(z)$ appearing in the introduction we first consider

$$J_{r,s}(z) = \iint\limits_{R_k} \frac{x^r y^s}{1 - xyz}\, dx\, dy, \tag{2.1}$$

where $r \in \mathbf{N}_0$, $s \in \mathbf{Z}$ and $R_k = (1, \zeta) \times (\zeta^{-1}, 1)$ is the rectangular region. Obviously (2.1) converges for $|z| < \zeta^{-1}$ as a real double improper integral. We then have

Lemma 2.1. *Each $J_{r,s}(z)$ is analytic in $\mathbf{C}$ with a branch cut $[\zeta^{-1}, \infty)$ and has the limit $I_{r,s}$ as z tends to 1 while remaining in the upper-half plane; more precisely,*

(i) *if $r = s \geq 0$, then*

$$I_{r,r} = -\frac{1}{2}\log^2 \zeta + \pi i \log \zeta - \sum_{\ell=1}^{r} \frac{\zeta^\ell - 2 + \zeta^{-\ell}}{\ell^2}; \tag{2.2}$$

(ii) *if $r \neq s \geq 0$, then*

$$I_{r,s} = \frac{1 - \zeta^{r-s}}{r - s}(\log \zeta - \pi i) + \sum_{\substack{\ell=-s \\ \ell \neq 0}}^{r} \frac{(\zeta^\ell - \zeta^{r-s})(1 - \zeta^{-\ell})}{(r-s)\ell}; \tag{2.3}$$

(iii) *if $r \geq 0 > s$, then*

$$I_{r,s} = \frac{\zeta^{r+|s|} - 1}{r + |s|}\pi i + \epsilon_{r,s} \sum_{\ell=\min(r,|s|-1)+1}^{\max(r,|s|-1)} \frac{(\zeta^\ell - \zeta^{r+|s|})(1 - \zeta^{-\ell})}{(r+|s|)\ell}, \tag{2.4}$$

where $\epsilon_{r,s} = 1, 0$ or -1 according as $r \geq |s|, r = |s| - 1$ or $r \leq |s| - 2$.

Proof. Let $L_1(z) = \sum_{\ell=1}^\infty z^\ell/\ell = -\operatorname{Log}(1-z)$ and $L_2(z) = \sum_{\ell=1}^\infty z^\ell/\ell^2$ be the dilogarithm, being analytic in $\mathbf{C}\backslash[1, \infty)$. We first consider the case $r = s \geq 0$ for $|z| < \zeta^{-1}$; then

$$\begin{aligned} z^{r+1} J_{r,r}(z) &= \sum_{\ell=r+1}^{\infty} \frac{\zeta^\ell - 2 + \zeta^{-\ell}}{\ell^2} z^\ell \\ &= L_2(z\zeta) - 2L_2(z) + L_2\left(\frac{z}{\zeta}\right) - \sum_{\ell=1}^{r} \frac{(z\zeta)^\ell - 2z^\ell + (z\zeta^{-1})^\ell}{\ell^2}, \end{aligned}$$

the right-hand side being analytic in $\mathbf{C}\backslash[\zeta^{-1}, \infty)$. Hence using the identity $L_2(z) + L_2(z^{-1}) = 2L_2(-1) - (\operatorname{Log}^2(-z))/2$ valid for $z \in \mathbf{C}\backslash[0, \infty)$. (See, for example, Levin's book [8, p.4].), we obtain

$$\begin{aligned} I_{r,r} = \lim_{\substack{z \to 1 \\ \operatorname{Im} z > 0}} J_{r,r}(z) &= \lim_{\substack{w \to \zeta \\ \operatorname{Im} w > 0}} \left(L_2(w) + L_2\left(\frac{1}{w}\right)\right) - 2L_2(1) - \sum_{\ell=1}^{r} \frac{\zeta^\ell - 2 + \zeta^{-\ell}}{\ell^2} \\ &= -\frac{1}{2}\log^2 \zeta + \pi i \log \zeta - \sum_{\ell=1}^{r} \frac{\zeta^\ell - 2 + \zeta^{-\ell}}{\ell^2}, \end{aligned}$$

as required. Similarly, for the case $r \neq s \geq 0$, we have

$$
\begin{aligned}
(r-s)z^{r+1}J_{r,s}(z) &= \sum_{\ell=0}^{\infty}\left(\frac{1}{s+\ell+1}-\frac{1}{r+\ell+1}\right)(\zeta^{r+\ell+1}-1)(1-\zeta^{-s-\ell-1})z^{r+\ell+1} \\
&= \left((z\zeta)^{r-s}-1\right)L_1(z\zeta)-(1+\zeta^{r-s})(z^{r-s}-1)L_1(z) \\
&\quad + (z^{r-s}-\zeta^{r-s})L_1\!\left(\frac{z}{\zeta}\right)+\sum_{\ell=1}^{r}\frac{(\zeta^{\ell}-\zeta^{r-s})(1-\zeta^{-\ell})}{\ell}z^{\ell} \\
&\quad - \sum_{\ell=1}^{s}\frac{(\zeta^{-\ell}-\zeta^{r-s})(1-\zeta^{\ell})}{\ell}z^{r-s+\ell},
\end{aligned}
$$

the right-hand side being analytic in $\mathbf{C}\backslash[\zeta^{-1},\infty)$. We thus get (2.3) by letting $z \to 1$ with $\mathrm{Im}(z) > 0$. Finally, for the case $r \geq 0 > s$, we have

$$
\begin{aligned}
(r+|s|)z^{r+1}J_{r,s}(z) &= z^{r+|s|}(\zeta^{r+|s|}-1)\log\zeta \\
&\quad + \sum_{\ell\neq|s|-1}\left(\frac{1}{-|s|+\ell+1}-\frac{1}{r+\ell+1}\right)(\zeta^{r+\ell+1}-1)(1-\zeta^{|s|-\ell-1})z^{r+\ell+1} \\
&= z^{r+|s|}(\zeta^{r+|s|}-1)\log\zeta+\left((z\zeta)^{r+|s|}-1\right)L_1(z\zeta) \\
&\quad - (1+\zeta^{r+|s|})(z^{r+|s|}-1)L_1(z)+(z^{r+|s|}-\zeta^{r+|s|})L_1\!\left(\frac{z}{\zeta}\right) \\
&\quad + \sum_{\ell=1}^{r}\frac{(\zeta^{\ell}-\zeta^{r+|s|})(1-\zeta^{-\ell})}{\ell}z^{\ell}-\sum_{\ell=1}^{|s|-1}\frac{(\zeta^{\ell}-\zeta^{r+|s|})(1-\zeta^{-\ell})}{\ell}z^{r+|s|-\ell},
\end{aligned}
$$

from which one has (2.4) by letting $z \to 1$ with $\mathrm{Im}(z) > 0$. □

We next consider the integral

$$J(z) = \iint\limits_{R_k} \frac{P(x)Q(y)}{1-xyz}\,dx\,dy,$$

where $P(x)$ and $y^L Q(y)$ are polynomials with integral coefficients of degrees N and $L+M$ respectively for positive integers L, M and N. It follows from Lemma 2.1 that $J(z)$ is analytic in $\mathbf{C}$ with a branch cut $[\zeta^{-1},\infty)$ and has the limit I as z tends to 1 while remaining in the upper-half plane. Let D_n be the least common multiple of $1, 2, ..., n$ and Δ_n be the product of all primes less than or equal to $\sqrt{2n}$. Note that $D_n^{1/n} \to e$ and $\Delta_n^{1/n} \to 1$ as $n \to \infty$ by the prime number theorem.

Lemma 2.2. *Suppose that* $\max\{L, M\} \leq N$ *and that* $P(x) = \sum_{j=0}^{N} \alpha_j x^j$ *and* $Q(y) = \sum_{j=-L}^{M} \beta_j y^j$ *satisfy the following three conditions*:

(i) *for* $0 \leq j < L$,

$$\int_1^{\zeta} x^j P(x)\,dx = 0;$$

(ii) $k^r|\alpha_r$ *for* $0 \leq r \leq N$, $(k+1)^s|\beta_s$ *for* $0 \leq s \leq M$ *and* $k^{|s|}|\beta_s$ *for* $-L \leq s \leq -1$;

(iii) *any prime lying in the interval* $(N, r+L]$ *is a divisor of* α_r *for each* $N-L < r \leq N$.

Then the limit I can be expressed in the form

$$I = a\left(\frac{1}{2}\log^2\zeta - \pi i \log\zeta\right) + b\big(\log\zeta - \pi i\big) + c \tag{2.5}$$

with $a \in \mathbf{Z}, b \in \mathbf{Z}/D_N$ and $c \in \mathbf{Z}/D_N^2\Delta_N$. Moreover we have

$$a = -\frac{1}{2\pi i}\int_C P(z)Q\left(\frac{1}{z}\right)\frac{dz}{z} \tag{2.6}$$

for any circle C centered at the origin.

Proof. Expanding $P(x)Q(y)$, we get immediately

$$I = \sum_{\substack{0\le r\le N\\ -L\le s\le M}} \alpha_r\beta_s\, I_{r,s}.$$

We first show that the corresponding sum of the coefficients of πi in the right-hand side of (2.4) vanishes; in fact,

$$\begin{aligned}\sum_{-L\le s<0\le r\le N} \alpha_r\beta_s \frac{\zeta^{r+|s|}-1}{r+|s|} &= \sum_{-L\le s<0}\beta_s \sum_{0\le r\le N}\alpha_r \int_1^\zeta x^{r+|s|-1}dx\\ &= \sum_{-L\le s<0}\beta_s\int_1^\zeta x^{|s|-1}P(x)\,dx = 0\end{aligned}$$

by the orthogonality condition (i). This means that the limit I can be expressed in the form (2.5) with some rational numbers a, b and c by virtue of (2.2), (2.3) and (2.4).

Obviously $a = -\sum_{r=0}^M \alpha_r\beta_r$ is an integer and

$$a = -\frac{1}{2\pi i}\int_C P(z)Q\left(\frac{1}{z}\right)\frac{dz}{z}$$

for any circle C centered at the origin. Since $\zeta^{r-s}\alpha_r \in \mathbf{Z}$ for $r > s \ge 0$ and $\zeta^{r-s}\beta_s \in \mathbf{Z}$ for $r < s$, it is easily seen that

$$b = \sum_{r\ne s\ge 0}\alpha_r\beta_s\frac{1-\zeta^{r-s}}{r-s} \in \frac{\mathbf{Z}}{D_N}.$$

Similarly we get

$$c' \equiv -\sum_{r=1}^M \alpha_r\beta_r\sum_{\ell=1}^r \frac{\zeta^\ell - 2 + \zeta^{-\ell}}{\ell^2} + \sum_{r\ne s\ge 0}\alpha_r\beta_s\sum_{\substack{\ell=-s\\ \ell\ne 0}}^r \frac{(\zeta^\ell - \zeta^{r-s})(1-\zeta^{-\ell})}{(r-s)\ell} \in \frac{\mathbf{Z}}{D_N^2},$$

since $\alpha_r\beta_s(\zeta^\ell + \zeta^{r-s-\ell}) \in \mathbf{Z}$ for $-s \le \ell \le r$. We finally consider

$$c'' \equiv \sum_{-L\le s<0\le r\le N}\epsilon_{r,s}\alpha_r\beta_s \sum_{\ell=\min(r,|s|-1)+1}^{\max(r,|s|-1)} \frac{(\zeta^\ell - \zeta^{r+|s|})(1-\zeta^{-\ell})}{(r+|s|)\ell},$$

belonging to $\mathbf{Z}/D_N D_{L+N}$ since $k^{r+|s|}|\alpha_r\beta_s$ for $r \ge 0 > s$. Now let p be any prime lying in $(N, L+N]$. If p is a prime factor of $(r+|s|)\ell$ for some ℓ, r and s, then

$p = r + |s|$ since $\ell \le N$ and $r + |s| \le L + N \le 2N$. Hence $p \le r + L$ and $p|\alpha_r$ by the divisibility condition (iii). This implies that

$$c'' \in \left(\prod_{\substack{p:\text{prime} \\ N<p\le L+N}} p \right) \frac{\mathbf{Z}}{D_N D_{L+N}}.$$

On the other hand,

$$\frac{D_{L+N}}{D_N} = \prod_{p:\text{prime}} p^{\nu(p)},$$

where $\nu(p) = [\log(L+N)/\log p] - [\log N/\log p]$. Note that $\nu(p)$ is either 0 or 1. Since $\nu(p) = 1$ for $p \in (N, L+N\,]$ and $\nu(p) = 0$ for $p \in (\sqrt{2N}, N\,]$, we obtain

$$\frac{D_{L+N}}{D_N} = \prod_{\substack{p:\text{prime} \\ p\le\sqrt{2N}}} p^{\nu(p)} \times \prod_{\substack{p:\text{prime} \\ N<p\le L+N}} p,$$

the first product on the right-hand side being a divisor of Δ_N. Therefore $c = c' + c'' \in \mathbf{Z}/D_N^2 \Delta_N$, which completes the proof. □

A sequence $\{x_n\}$ is said to be *eventually stationary in* $\mathbf{Q}$ provided that there exist a $\kappa \in \mathbf{Q}$ and $n_0 \in \mathbf{N}$ satisfying $x_n = \kappa$ for all $n \ge n_0$. We need the following lemma concerning a non-quadraticity measure, which is a corollary of [5, Lemma 2.1].

Lemma 2.3. *Let* γ *be a real number satisfying*

$$q_n\gamma - p_n = \varepsilon_n \qquad \text{and} \qquad q_n\gamma^2 - r_n = \delta_n$$

where $p_n, q_n, r_n \in \mathbf{Z}$ *with* $q_n \ne 0$ *for all* $n \ge 1$. *Suppose that*

$$\lim_{n\to\infty} \frac{1}{n}\log|q_n| = \sigma \quad \text{and} \quad \max\left\{ \lim_{n\to\infty} \frac{1}{n}\log|\varepsilon_n|, \quad \lim_{n\to\infty} \frac{1}{n}\log|\delta_n| \right\} = -\tau \tag{2.7}$$

for positive numbers σ *and* τ. *Suppose further that the sequence* $\{\delta_n/\varepsilon_n\}$ *is not eventually stationary in* $\mathbf{Q}$. *Then the number* γ *has a non-quadraticity measure* $1 + \sigma/\tau$; *more precisely, for any* $\varepsilon > 0$, *there exists a positive constant* $H_0(\varepsilon)$ *such that*

$$\bigl|\gamma - \xi\bigr| \ge H^{-1-\sigma/\tau-\varepsilon} \tag{2.8}$$

for any quadratic number ξ *with* $H \equiv H(\xi) \ge H_0(\varepsilon)$. *Moreover, if the limits in* (2.7) *are calculated effectively, then the above constant* $H_0(\varepsilon)$ *is also effectively computable.*

Remark 1. If we know beforehand the irrationality of γ, then the conditions in (2.7) concerning the remainder terms ε_n and δ_n can be weakened as follows:

$$\max\left\{ \limsup_{n\to\infty} \frac{1}{n}\log|\varepsilon_n|, \quad \limsup_{n\to\infty} \frac{1}{n}\log|\delta_n| \right\} \le -\tau.$$

If we know, in addition, that γ is not quadratic, then the sequence $\{\delta_n/\varepsilon_n\}$ must be not eventually stationary in $\mathbf{Q}$. This is our case since $\gamma = \log\zeta$ is transcendental. (2.8) can be easily obtained from the lower estimate $|p + q\gamma + r\gamma^2| \ge H^{-\sigma/\tau-\varepsilon'}$ by standard argument.

3. Simultaneous approximations

To construct explicitly simultaneous rational approximations to $\log\zeta$ and $\log^2\zeta$ using Lemma 2.2, we define

$$P_n(x) = k^{Bn}\frac{(k+1-kx)^{Bn}}{(An)!}\left((x-1)^{(A+B)n}(k+1-kx)^{(A-B)n}\right)^{(An)}$$

and

$$Q_n(y) = \left(\frac{1}{y}-1\right)^{(A-B)n}((k+1)y-k)^{(A+B)n},$$

for any $n \in \mathbf{N}$, where A, B are arbitrarily fixed coprime integers with $A > B \geq 1$.

We first show that $P_n(x)$ and $Q_n(y)$ satisfy the conditions stated in Lemma 2.2 with $L = (A-B)n$ and $M = N = (A+B)n$. Indeed the orthogonality condition (i) is clearly fulfilled by partial integration. Then, putting $P_n(x) = \sum_{r=0}^{N}\alpha_r x^r$, we get

$$\alpha_r = (-1)^{Bn+r}\sum_{\substack{\ell_1+\ell_2=r\\ 0\leq\ell_1\leq Bn\\ 0\leq\ell_2\leq An}}\binom{Bn}{\ell_1}\binom{An+\ell_2}{An}$$
$$\times\sum_{\substack{\ell_3+\ell_4=An+\ell_2\\ 0\leq\ell_3\leq(A-B)n\\ 0\leq\ell_4\leq(A+B)n}} k^{Bn+\ell_1+\ell_3}(k+1)^{An-\ell_1-\ell_3}\binom{(A-B)n}{\ell_3}\binom{(A+B)n}{\ell_4};$$

hence $k^r|\alpha_r$ since $Bn+\ell_1+\ell_3 = (A+B)n+\ell_1+\ell_2-\ell_4 \geq \ell_1+\ell_2 = r$. Similarly, putting $Q_n(y) = \sum_{s=-L}^{M}\beta_s y^s$,

$$\beta_s = (-1)^s\sum_{\substack{\ell_6-\ell_5=s\\ 0\leq\ell_5\leq(A-B)n\\ 0\leq\ell_6\leq(A+B)n}} k^{(A+B)n-\ell_6}(k+1)^{\ell_6}\binom{(A-B)n}{\ell_5}\binom{(A+B)n}{\ell_6};$$

thus $(k+1)^s|\beta_s$ for $s \geq 0$ since $\ell_6 = \ell_5 + s \geq s$ and $k^{|s|}|\beta_s$ for $s < 0$ since $(A+B)n - \ell_6 = (A+B)n - \ell_5 + |s| > |s|$. Finally let p be any prime lying in $((A+B)n, (A-B)n+r]$ for $r \in (2Bn, (A+B)n]$. Since $p > An \geq \ell_2$ and $p \leq (A-B)n + r = (A-B)n+\ell_1+\ell_2 \leq An+\ell_2$, it follows that p is a prime factor of the binomial coefficient $\binom{An+\ell_2}{An}$ for any ℓ_2 with $\ell_1+\ell_2 = r$; hence $p|\alpha_r$, as required.

Thus, applying Lemma 2.2, the limit I_n of the integral

$$J_n(z) = \iint\limits_{R_k}\frac{P_n(x)Q_n(y)}{1-xyz}dx\,dy,$$

as z tends to 1 with $\mathrm{Im}(z) > 0$, can be expressed in the form

$$I_n = a_n\left(\frac{1}{2}\log^2\zeta - \pi i\log\zeta\right) + b_n\big(\log\zeta - \pi i\big) + c_n, \tag{3.1}$$

where $a_n \in \mathbf{Z}$, $b_n \in \mathbf{Z}/D_{(A+B)n}$ and $c_n \in \mathbf{Z}/D^2_{(A+B)n}\Delta_{(A+B)n}$. We hence have simultaneous rational approximations to $\log\zeta$ and $\log^2\zeta$ by taking the real and imaginary parts of (3.1), as follows:

$$\begin{cases} a_n \log\zeta + b_n = -\dfrac{1}{\pi}\operatorname{Im}(I_n), \\ a_n \log^2\zeta - 2c_n = -2\operatorname{Re}(I_n) - \dfrac{2\log\zeta}{\pi}\operatorname{Im}(I_n). \end{cases} \tag{3.2}$$

We next investigate the denominators of the rationals b_n and c_n. For any prime $p > \sqrt{2An}$, put $\omega = \{An/p\}$, $\eta = \{Bn/p\}$ and $\theta_j = \{\ell_j/p\}$ for $2 \le j \le 4$, where $\{x\}$ denotes the fractional part of x. Suppose that $p \nmid \alpha_r$ for some $r \in [0, (A+B)n]$. Then p is not a divisor of $\binom{An+\ell_2}{An}\binom{(A-B)n}{\ell_3}\binom{(A+B)n}{\ell_4}$ for some ℓ_2, ℓ_3 and ℓ_4; hence $[\omega+\theta_2] = 0$, $[\omega-\eta] = [\omega-\eta-\theta_3]$ and $[\omega+\eta] = [\omega+\eta-\theta_4]$ for some θ_2, θ_3 and θ_4. This implies that $\omega+\theta_2 < 1$, $\{\omega-\eta\} \ge \theta_3$ and $\{\omega+\eta\} \ge \theta_4$. Since $\theta_3+\theta_4 \equiv \omega+\theta_2 \pmod 1$, it follows that

$$\omega \le \omega+\theta_2 \le \theta_3+\theta_4 \le \{\omega+\eta\}+\{\omega-\eta\}.$$

This means that if $p > \sqrt{2An}$ satisfies $\omega > \{\omega+\eta\}+\{\omega-\eta\}$, then p becomes a common factor of all the coefficients of $P_n(x)$. Put $\tilde{P}_n(x) \equiv P_n(x)/\Lambda_n \in \mathbf{Z}[x]$ where

$$\Lambda_n = \prod_{\substack{p:\text{prime} \\ p>\sqrt{2An} \\ \omega>\{\omega+\eta\}+\{\omega-\eta\}}} p.$$

Note that any prime p satisfying $\omega > \{\omega+\eta\}+\{\omega-\eta\}$ is less than or equal to $(A+B)n$. Thus the polynomial $\tilde{P}_n(x)$ also satisfies the conditions in Lemma 2.2; hence we get

$$b_n \in \frac{\mathbf{Z}}{D_{(A+B)n}/\Lambda_n} \quad \text{and} \quad c_n \in \frac{\mathbf{Z}}{(D_{(A+B)n}/\Lambda_n)D_{(A+B)n}\Delta_{(A+B)n}}. \tag{3.3}$$

4. Another integral representation

We can obtain further arithmetical information on the rationals b_n and c_n by using another integral representation of I_n. To see this we need two lemmas as follows.

Lemma 4.1. *For $d > 0$ and $\theta \in (0, \pi/2)$ let $S_{d,\theta}$ be the angular sector formed by the points $z = d + te^{i\varphi}$ with $t > 0$ and $\theta < \varphi < \pi - \theta$. For any $\sigma, \tau \ge 0$ satisfying $\sigma + \tau > 1$, the function*

$$\frac{|u-d|^\sigma |v-d^{-1}|^\tau}{|1-uv|}$$

converges to 0 as u, v tend to d, d^{-1} while remaining in the angular sectors $S_{d,\theta}$, $S_{d^{-1},\theta}$ respectively.

Proof. Put $u = d + \varepsilon e^{i\varphi}$ and $v = d^{-1} + \varepsilon' e^{i\varphi'}$ for $\varepsilon, \varepsilon' > 0$ and $\varphi, \varphi' \in (\theta, \pi - \theta)$. We can assume that $\varepsilon' \le d^* = \frac{1}{2}\min(d, d^{-1}) \sin\theta$. Then

$$\begin{aligned}|1 - uv| &= |d\varepsilon' e^{i\varphi'} + d^{-1}\varepsilon e^{i\varphi} + \varepsilon\varepsilon' e^{i(\varphi+\varphi')}| \\ &> (d\varepsilon' + d^{-1}\varepsilon)\sin\theta - \varepsilon\varepsilon' > d^*(\varepsilon + \varepsilon');\end{aligned}$$

therefore

$$\frac{|u-d|^\sigma |v - d^{-1}|^\tau}{|1-uv|} < \frac{1}{d^*} \cdot \frac{\varepsilon^\sigma \varepsilon'^\tau}{\varepsilon + \varepsilon'}.$$

The right-hand side clearly converges to 0 as $\varepsilon, \varepsilon' \to 0+$, since $\sigma + \tau > 1$. □

Lemma 4.2. *Let $P(x)$ and $y^L Q(y)$ be polynomials for some $L \in \mathbf{N}$ with $P(1) = P(\zeta) = Q(1) = Q(\zeta^{-1}) = 0$. Then the limit of the integral*

$$J(z) = \iint_{R_k} \frac{P(x)Q(y)}{1 - xyz}\, dx\, dy,$$

as z tends to 1 with $\mathrm{Im}(z) > 0$, can be expressed in the form

$$\iint_{\alpha\times\beta} \frac{P(u)Q(v)}{1-uv}\, du\, dv,$$

where $\alpha, \beta : (-1, 1) \to \mathbf{C}$ are any rectifiable paths lying in the upper-half plane satisfying the following boundary conditions for some $\theta \in (0, \pi/2)$ and $s_0 \in (0, 1)$:

(i) *$\alpha(s) \in S_{1,\theta}, \beta(s) \in S_{\zeta^{-1},\theta}$ for $s \in (-1, -1 + s_0]$ and $\alpha(s) \to 1, \beta(s) \to \zeta^{-1}$ as $s \to -1$;*

(ii) *$\alpha(s) \in S_{\zeta,\theta}, \beta(s) \in S_{1,\theta}$ for $s \in [1 - s_0, 1)$ and $\alpha(s) \to \zeta, \beta(s) \to 1$ as $s \to 1$.*

Proof. We first consider the paths α, β satisfying $\mathrm{Im}(\alpha(s)\beta(t)) > 0$ for any $(s, t) \in (-1, 1)^2$. For example, the upper semicircles centered at $(1 + \zeta)/2, (1 + \zeta^{-1})/2$ with radii $(\zeta - 1)/2, (1 - \zeta^{-1})/2$ respectively, satisfy this condition. Since α, β are bounded, it follows from Cauchy's theorem that

$$J(z) = \iint_{\alpha\times\beta} \frac{P(u)Q(v)}{1 - uvz}\, du\, dv,$$

for sufficiently small $|z|$, the right-hand side being analytic in the upper-half plane. On the other hand, we know that $J(z)$ is analytic in $\mathbf{C}$ with a branch cut $[\zeta^{-1}, \infty)$ and has the limit I as z tends to 1 with $\mathrm{Im}(z) > 0$; in particular,

$$I = \lim_{\varepsilon\to 0+} J(1 + \varepsilon i) = \lim_{\varepsilon\to 0+} \iint_{\alpha\times\beta} \frac{P(u)Q(v)}{1 - uv(1 + \varepsilon i)}\, du\, dv.$$

Since $\mathrm{Im}(uv) > 0$ for any $(u, v) \in \alpha\times\beta$, it is easily seen that $|1-uv(1+\varepsilon i)| \ge |1-uv|$ for any $\varepsilon > 0$; therefore

$$\left|\frac{P(u)Q(v)}{1 - uv(1 + \varepsilon i)}\right| \le const. \left|\frac{(u-1)(u-\zeta)(v-1)(v-\zeta^{-1})}{1-uv}\right|. \tag{4.1}$$

Since $uv \to 1$ if and only if either $(u, v) \to (1, 1)$ or $(u, v) \to (\zeta, \zeta^{-1})$, it follows from Lemma 4.1 that the right-hand side of (4.1) is bounded on $\alpha \times \beta$. Hence, by

Lebesgue's convergence theorem, we get

$$I = \iint\limits_{\alpha\times\beta} \frac{P(u)Q(v)}{1-uv}\, du\, dv,$$

as required.

Finally Cauchy's theorem allows us to change the specific paths discussed above to arbitrary rectifiable paths lying in the upper-half plane and satisfying the boundary conditions stated in the lemma. This completes the proof. □

Thus, applying Lemma 4.2 to $P_n(x)$ and $Q_n(y)$, we get

$$I_n = \iint\limits_{\alpha\times\beta} \frac{P_n(u)Q_n(v)}{1-uv}\, du\, dv.$$

Then An-fold partial integration with respect to u, together with repeated application of Lemma 4.1, implies that

(4.2)

$$\begin{aligned} I_n &= (-1)^{An}\frac{k^{(A+B)n}}{(An)!}\iint\limits_{\alpha\times\beta}\left((u-1)^{A+B}(\zeta-u)^{A-B}\right)^n\left(\frac{(\zeta-u)^{Bn}}{1-uv}\right)^{(An)}Q_n(v)\, du\, dv \\ &= (-k)^{An}(k+1)^{(A+2B)n} \\ &\qquad \times\iint\limits_{\alpha\times\beta}\left(\frac{(u-1)^{A+B}(\zeta-u)^{A-B}(1-v)^{A-B}(v-\zeta^{-1})^{A+2B}}{(1-uv)^A}\right)^n\frac{du\, dv}{1-uv}. \end{aligned}$$

The last expression will also be used to obtain the principal part of I_n applying $\mathbf{C}^2$-saddle method. Now, substituting $U=\zeta v$ and $V=\zeta^{-1}u$, it follows that

(4.3)

$$\begin{aligned} I_n &= (-1)^{An}\big(k(k+1)\big)^{(A+B)n} \\ &\qquad \times\iint\limits_{\tilde\alpha\times\tilde\beta}\left(\frac{(U-1)^{A+2B}(\zeta-U)^{A-B}(1-V)^{A-B}(V-\zeta^{-1})^{A+B}}{(1-UV)^A}\right)^n\frac{dU\, dV}{1-UV} \end{aligned}$$

where $\tilde\alpha=\zeta\beta$ and $\tilde\beta=\zeta^{-1}\alpha$. Obviously $\tilde\alpha$ and $\tilde\beta$ satisfy the boundary conditions in Lemma 4.2 with the same θ; hence $\tilde\alpha\times\tilde\beta$ in (4.3) can be replaced by $\alpha\times\beta$.

On the other hand, we introduce another integral

$$J_n^*(z) = \iint\limits_{R_k} \frac{P_n^*(x)Q_n^*(y)}{1-xyz}\, dx\, dy$$

with

$$P_n^*(x) = k^{2Bn}\frac{(k+1-kx)^{Bn}}{(An)!}\left((x-1)^{(A+2B)n}(k+1-kx)^{(A-B)n}\right)^{(An)}$$

and

$$Q_n^*(y) = \left(\frac{1}{y}-1\right)^{(A-B)n}((k+1)y-k)^{An}.$$

Then it can be seen that $P_n^*(x)$ and $Q_n^*(y)$ satisfy the conditions in Lemma 2.2 with $L = (A-B)n$, $M = An$ and $N = (A+2B)n$. Applying Lemmas 2.2 and 4.2 to $P_n^*(x)$ and $Q_n^*(y)$, it follows that the limit

$$I_n^* = \lim_{\substack{z\to 1\\ \operatorname{Im} z>0}} J_n^*(z) = \iint\limits_{\alpha\times\beta} \frac{P_n^*(u)Q_n^*(v)}{1-uv}\,du\,dv$$

can be expressed in the form

$$I_n^* = a_n^*\left(\frac{1}{2}\log^2\zeta - \pi i \log\zeta\right) + b_n^*\big(\log\zeta - \pi i\big) + c_n^*$$

with $a_n^* \in \mathbf{Z}$, $b_n^* \in \mathbf{Z}/D_{(A+2B)n}$ and $c_n^* \in \mathbf{Z}/D_{(A+2B)n}^2\Delta_{(A+2B)n}$. After An-fold partial integration with respect to u together with repeated application of Lemma 4.1, one has

$$I_n^* = (-1)^{An}\big(k(k+1)\big)^{(A+B)n} \times \iint\limits_{\alpha\times\beta}\left(\frac{(u-1)^{A+2B}(\zeta-u)^{A-B}(1-v)^{A-B}(v-\zeta^{-1})^{A+B}}{(1-uv)^A}\right)^n \frac{du\,dv}{1-uv}.$$

Comparing with (4.3), we get $I_n = I_n^*$; therefore $a_n = a_n^*$, $b_n = b_n^*$ and $c_n = c_n^*$ by the irrationality of $\log\zeta$.

Now, putting $P_n^*(x) = \sum_{r=0}^N \alpha_r^* x^r$, we have

$$\alpha_r^* = (-1)^r \sum_{\substack{\ell_1+\ell_2=r\\ 0\le\ell_1\le Bn\\ 0\le\ell_2\le(A+B)n}} \binom{Bn}{\ell_1}\binom{An+\ell_2}{An} \times \sum_{\substack{\ell_3+\ell_4=An+\ell_2\\ 0\le\ell_3\le(A-B)n\\ 0\le\ell_4\le(A+2B)n}} k^{2Bn+\ell_1+\ell_3}(k+1)^{An-\ell_1-\ell_3}\binom{(A-B)n}{\ell_3}\binom{(A+2B)n}{\ell_4}.$$

Suppose that a prime $p > \sqrt{3An}$ is not a divisor of α_r^* for some $r \in [0,(A+2B)n]$. Then $p \nmid \binom{An+\ell_2}{An}\binom{(A-B)n}{\ell_3}\binom{(A+2B)n}{\ell_4}$ for some ℓ_2, ℓ_3 and ℓ_4; hence $[\omega+\theta_2]=0$, $[\omega-\eta] = [\omega-\eta-\theta_3]$ and $[\omega+2\eta] = [\omega+2\eta-\theta_4]$ for some θ_2, θ_3 and θ_4. This implies that $\omega+\theta_2<1$, $\{\omega-\eta\}\ge\theta_3$ and $\{\omega+2\eta\}\ge\theta_4$; thus

$$\omega \le \omega+\theta_2 \le \theta_3+\theta_4 \le \{\omega+2\eta\}+\{\omega-\eta\},$$

since $\theta_3+\theta_4 \equiv \omega+\theta_2 \pmod 1$. This means that if $p>\sqrt{3An}$ satisfies $\omega > \{\omega+2\eta\}+\{\omega-\eta\}$, then $p|\alpha_r^*$ for all r. Hence we have

$$b_n^* \in \frac{\mathbf{Z}}{D_{(A+2B)n}/\Lambda_n^*} \quad\text{and}\quad c_n^* \in \frac{\mathbf{Z}}{(D_{(A+2B)n}/\Lambda_n^*)D_{(A+2B)n}\Delta_{(A+2B)n}} \tag{4.4}$$

where Λ_n^* is the product of all primes greater than $\sqrt{3An}$ and satisfying $\omega > \{\omega+2\eta\}+\{\omega-\eta\}$.

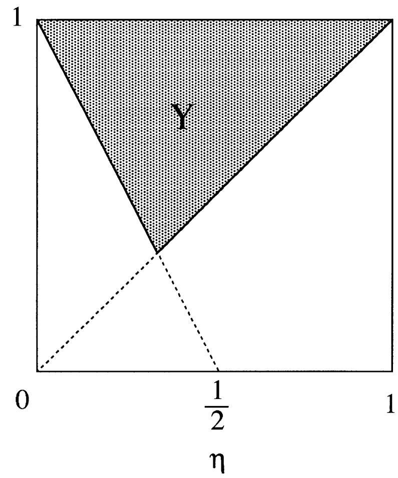

FIGURE 2.

Combining (3.3) and (4.4), we conclude that

$$b_n \in \frac{\mathbf{Z}}{D_{(A+B)n}/\Lambda_n^{**}} \quad \text{and} \quad c_n \in \frac{\mathbf{Z}}{(D_{(A+B)n}/\Lambda_n^{**})D_{(A+B)n}\Delta_{(A+B)n}}$$

where Λ_n^{**} is the product of all primes $p \in (\sqrt{3An}, (A+B)n]$ satisfying

$$\omega > \{\omega - \eta\} + \min\big\{\{\omega + \eta\}, \{\omega + 2\eta\}\big\}. \tag{4.5}$$

Put $M_n = (D_{(A+B)n}/\Lambda_n^{**})D_{(A+B)n}\Delta_{(A+B)n}$ for brevity. We then have from (3.2)

$$\begin{cases} a_n M_n \log\zeta + b_n M_n = -\dfrac{M_n}{\pi}\operatorname{Im}(I_n), \\ a_n M_n \log^2\zeta - 2c_n M_n = -2M_n\left(\operatorname{Re}(I_n) - \dfrac{\log\zeta}{\pi}\operatorname{Im}(I_n)\right), \end{cases} \tag{4.6}$$

with some integers $a_n M_n$, $b_n M_n$ and $c_n M_n$.

Let Y be the set of points $(\eta, \omega) \in (0,1)^2$ satisfying (4.5); namely, let $Y = \{(\eta,\omega) \in (0,1)^2; \omega > \max\{\eta, 1-2\eta\}\}$ (Figure 2). Then the asymptotic behavior of Λ_n^{**}, as n tends to infinity, can be easily obtained by the same way as in [6, Lemma 3.1], as follows:

$$\chi(A,B) \equiv \lim_{n\to\infty}\frac{1}{n}\log\Lambda_n^{**} = \int_{Z_{A,B}} \frac{dx}{x^2} \tag{4.7}$$

where $Z_{A,B} = \{x > (A+B)^{-1}; (\{Bx\},\{Ax\}) \in Y\}$. Moreover the integral in (4.7) can be expressed as a finite combination of values of elementary functions, which takes a fairly complicated form in general. However, in the case $B = 1$, $\chi(A) \equiv \chi(A,1)$ can be expressed in a comparatively simple form

$$\begin{aligned} \chi(A) &= (A-1)\log(A-1) - A\log A - 1 \\ &\quad - \sum_{\ell=1}^{[(A+2)/3]}\psi_0\left(\frac{\ell}{A+2}\right) + \sum_{\ell=2}^{[(A+2)/3]}\psi_0\left(\frac{\ell-1}{A-1}\right) \end{aligned} \tag{4.8}$$

where $\psi_0(x) = \psi(x) - \psi(1)$ and $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function.

5. Saddles of rational functions

This section is devoted to studying several basic properties on saddles of rational functions, which will be used in application of the $\mathbf{C}^2$-saddle method in Section 6.

Let $R(z)$ be a non-constant rational function and z_0 be a saddle of $R(z)$. For any open set $X \subset \mathbf{C}$ and any point $z \in \partial X$, we denote by $i(X; z)$ the number of connected components K of X satisfying $z \in \partial K$. We call $i(X; z)$ the *index* of X at z. If the limit

$$i_{\ell oc}(X; z_0) = \lim_{\varepsilon \to 0+} i(X \cap N_\varepsilon(z_0); z_0)$$

exists, we call it the *local index* of X at z_0. Put $n(R; z_0) = i(V(R; z_0); z_0)$ and $m(R; z_0) = i(U(R; z_0); z_0)$ for brevity, where $U(R; z_0)$ is the union of all connected components H of $\mathbf{C}\backslash\overline{V(R; z_0)}$ satisfying $z_0 \in \partial H$. Note that each bounded connected component K of $V(R; z_0)$ is a region whose boundary is a piecewise-continuously differentiable closed curve; so any distinct points $w_1, w_2 \in \overline{K}$ can be joined by a smooth path $\gamma : (0,1) \to K$ with $\gamma(0+) = w_1, \gamma(1-) = w_2$. The same thing holds for $U(R; z_0)$.

Clearly $n(R; z_0) \leq \mathrm{ord}(R; z_0)$ and $m(R; z_0) \leq \mathrm{ord}(R; z_0)$. Concerning a lower bound of the sum $n(R; z_0) + m(R; z_0)$ we generally have

Lemma 5.1. $n(R; z_0) + m(R; z_0) \geq 1 + \mathrm{ord}(R; z_0)$.

Proof. Let $K_1, K_2, ..., K_n$ be connected components of $V(R; z_0)$ where $n \equiv n(R; z_0)$. Put $i_j = i_{\ell oc}(K_j; z_0)$ for brevity. Obviously

$$\sum_{j=1}^{n} i_j = \mathrm{ord}(R; z_0).$$

Note that each K_j surrounds exactly $i_j - 1$ bounded connected components H of $U(R; z_0)$ satisfying $\partial H \subset \partial K_j$. Thus the total number of such connected components of $U(R; z_0)$ is

$$\sum_{j=1}^{n} (i_j - 1) = \mathrm{ord}(R; z_0) - n(R; z_0).$$

By noticing that there exists at least one connected component of $U(R; z_0)$ which is not counted in the above manner, we get

$$m(R; z_0) \geq 1 + \sum_{j=1}^{n} (i_j - 1) = 1 + \mathrm{ord}(R; z_0) - n(R; z_0),$$

as required. □

Lemma 5.2. *Any bounded connected component K of $V(R; z_0)$ contains at least one zero point of $R(z)$. Any bounded connected component H of $U(R; z_0)$ contains at least one pole of $R(z)$.*

Proof. Suppose, on the contrary, that $R(z) \neq 0$ on K. Then $1/R(z)$ is analytic on $\bar{K}$ and $1/|R(z)| = 1/|R(z_0)|$ on ∂K. By the maximum principle it follows that $|R(z)| \geq |R(z_0)|$ on K. This contradicts the definition of the valley set.

Similarly suppose that $R(z)$ is analytic on $\overline{H}$. Since $|R(z)| = |R(z_0)|$ on ∂H, it follows from the maximum principle that $|R(z)| \leq |R(z_0)|$ on H. On the other hand, it is easily seen that there exists a point $z' \in H$ satisfying $|R(z')| > |R(z_0)|$ since $H \cap V(R; z_0) = \varnothing$ and $\partial H \subset \partial V(R; z_0)$. This contradiction completes the proof. □

Suppose that $R(z)$ has zero points $u_1, u_2, ..., u_r$ and finite poles $v_1, v_2, ..., v_s$ with counting the multiplicities for $r > s \geq 0$; that is, $R(z)$ can be written as

$$R(z) = c\,\frac{\prod_{j=1}^{r}(z - u_j)}{\prod_{\ell=1}^{s}(z - v_\ell)}$$

with a non-zero complex constant c. We then have

Lemma 5.3. *Assume that $u_j \in \mathbf{R}$ for $1 \leq j \leq r$, $\mathrm{Im}(v_\ell) \leq 0$ for $1 \leq \ell \leq s$ and that $\mathrm{Im}(v_n) < 0$ for some $n \in [1, s]$. Put $a(x, y) = |R(x + iy)|$ for $x, y \in \mathbf{R}$ with $x + iy \neq v_\ell$ for any $1 \leq \ell \leq s$. Then $\frac{\partial a}{\partial y}(x, 0) < 0$ for any x except for zeros and real poles of $R(z)$. In particular, there are no saddles on the real axis.*

Proof. Since

$$\frac{\frac{\partial a}{\partial y}(x, y)}{a(x, y)} = \sum_{j=1}^{r} \frac{y}{|x + iy - u_j|^2} - \sum_{\ell=1}^{s} \frac{y - \mathrm{Im}(v_\ell)}{|x + iy - v_\ell|^2},$$

it follows that

$$\frac{\partial a}{\partial y}(x, 0) = |R(x)| \sum_{\ell=1}^{s} \frac{\mathrm{Im}(v_\ell)}{|x - v_\ell|^2} < 0.$$

□

From now on we consider a rational function $R(z)$ satisfying the conditions stated in Lemma 5.3. Then any connected component of $V(R; z_0)$ is bounded; hence $V(R; z_0)$ contains at least $n(R; z_0)$ zero points of $R(z)$ by Lemma 5.2. Since $U(R; z_0)$ contains at most one unbounded connected component, at least $m(R; z_0) - 1$ finite poles of $R(z)$ lie in $U(R; z_0)$. Moreover $\partial V(R; z_0)\backslash\{z_0\}$ contains at least one saddle of $R(z)$ if $n(R; z_0) + m(R; z_0) > 1 + \mathrm{ord}(R; z_0)$.

For any subset $X \subset \mathbf{C}$ put $X_+ = \{z \in X; \mathrm{Im}(z) > 0\}$, $X_- = \{z \in X; \mathrm{Im}(z) < 0\}$ and $X_{\mathbf{R}} = X \cap \mathbf{R}$ for brevity. Also let $r(X)$ be the reflected image of X with respect to the real axis; that is, $r(X) = \{z \in \mathbf{C}; \bar{z} \in X\}$. Since the real axis crosses any connected component K of $V(R; z_0)$, K is decomposed into three parts as $K = K_+ \cup K_{\mathbf{R}} \cup K_-$. Then we have

Lemma 5.4. $r((\overline{K})_-) \subset K_+$.

Proof. For any $z \in (\overline{K})_-$ we have $|\bar{z} - u_j| = |z - u_j| > 0$, $|\bar{z} - v_\ell| \geq |z - v_\ell| > 0$ for any j, ℓ and $|\bar{z} - v_n| > |z - v_n|$ for some $n \in [1, s]$; therefore $|R(\bar{z})| < |R(z)| \leq |R(z_0)|$. Since the set $r((\overline{K})_-)$ touches K_- on the real axis in the same way as K_+, one gets $r((\overline{K})_-) \subset K_+$. This completes the proof. □

Lemma 5.5. *For any saddle z_0 of $R(z)$ and for any connected component K of $V(R; z_0)$, the set K_+ is connected.*

Proof. For any distinct points w_1, w_2 in K_+ take a path $\gamma : [0,1] \to K$ with $\gamma(0) = w_1$ and $\gamma(1) = w_2$. Put

$$\gamma^*(s) = \begin{cases} \gamma(s) & \text{if } \gamma(s) \in K_+ \cup K_{\mathbf{R}}, \\ \overline{\gamma(s)} & \text{if } \gamma(s) \in K_-. \end{cases}$$

Then obviously γ^* is a path joining w_1 and w_2 with the same length as γ. Since $\gamma^*(s) \in r(K_-) \subset K_+$ for any $\gamma(s) \in K_-$ by Lemma 5.4, we have $\gamma^* \subset K_+ \cup K_{\mathbf{R}}$. Now let $N_\varepsilon(X)$ be the ε-neighborhood of $X \subset \mathbf{C}$ for $\varepsilon > 0$. Since $\gamma^* \cap K_{\mathbf{R}}$ is a compact subset of K, $N_\varepsilon(\gamma^* \cap K_{\mathbf{R}}) \subset K$ for a sufficiently small $\varepsilon > 0$. One can thus obtain a new path $\gamma^{**} : [0,1] \to K_+$ with $\gamma^{**}(0) = w_1$ and $\gamma^{**}(1) = w_2$ by modifying γ^* slightly in a small neighborhood of $\gamma^* \cap K_{\mathbf{R}}$. The set K_+ is therefore connected. □

We finally determine the numbers $m(R; z_0)$ and $n(R; z_0)$ for any saddle z_0 of $R(z)$.

Lemma 5.6. *For any saddle z_0 in the upper-half plane we have $m(R; z_0) = 1$ and $n(R; z_0) = \mathrm{ord}(R; z_0)$. Conversely $m(R; z_0) = \mathrm{ord}(R; z_0)$ and $n(R; z_0) = 1$ for any saddle z_0 in the lower-half plane.*

Proof. We first consider any saddle z_0 in the lower-half plane. Suppose, on the contrary, that $n(R; z_0) \geq 2$. Let K^0 and K^1 be distinct connected components of $V(R; z_0)$. Since $z_0 \in \big(\overline{K^0}\big)_- \cap \big(\overline{K^1}\big)_-$, it follows from Lemma 5.4 that

$$\overline{z_0} \in r\big(\big(\overline{K^0}\big)_-\big) \cap r\big(\big(\overline{K^1}\big)_-\big) \subset K^0_+ \cap K^1_+,$$

contrary to the fact that $K^0 \cap K^1 = \varnothing$. Thus $n(R; z_0) = 1$ and hence $m(R; z_0) = \mathrm{ord}(R; z_0)$ by Lemma 5.1.

We next consider any saddle z_0 in the upper-half plane. Suppose, on the contrary, that $m(R; z_0) \geq 2$. Let H^0 and H^1 be distinct connected components of $U(R; z_0)$. We can assume that H^0 is bounded. The real axis crosses H^0 since $z_0 \in \partial H$ and since H^0 contains at least one pole whose imaginary part is not positive. Then the boundary $B = \partial H^0$, being a piecewise-continuously differentiable closed curve, is decomposed into three parts as $B_+ \cup B_{\mathbf{R}} \cup B_-$. Put $B_0 = B_+ \cup B_{\mathbf{R}} \cup r(B_-)$. Obviously B_0 is a continuous closed curve. Let $K^1, K^2, ..., K^\ell$ be all connected components of $V(R; z_0)$ satisfying $\partial K^j \cap B_- \neq \varnothing$. It then follows from Lemma 5.4 that

$$r(B_-) \subset \bigcup_{j=1}^{\ell} r\big(\big(\overline{K^j}\big)_-\big) \subset \bigcup_{j=1}^{\ell} K^j_+;$$

hence $B_+ \cap r(B_-) = \varnothing$. Therefore B_0 encloses some region, say W, satisfying $W \cap H = \varnothing$. This implies that W includes another H^1, contrary to the fact that H^1 contains at least one pole of $R(z)$. Thus $m(R; z_0) = 1$ and hence $n(R; z_0) = \mathrm{ord}(R; z_0)$ by Lemma 5.1. This completes the proof. □

Remark 2. Lemma 5.5 still holds for any rational function $R(z)$ whose zero points and poles are all real, since $r(K_-) = K_+$ by the symmetry of level curves with respect to the real axis. In consequence $n(R; z_0) = \mathrm{ord}(R; z_0)$ for any saddle z_0 in the upper-half plane. To see this suppose, on the contrary, that there exists a connected component K of $V(R; z_0)$ with $i_{\ell oc}(K; z_0) \geq 2$. Since K_+ is connected, K_+ must enclose some connected component H of $U(R; z_0)$, a contradiction.

6. Application of $\mathbf{C}^2$-saddle method

We are now ready to apply our $\mathbf{C}^2$-saddle method to the integral

$$\tilde{I}_n = \iint_{\alpha\times\beta} g(w,z)\big(f(w,z)\big)^n\, dw\, dz$$

where $g(w,z) = 1/(1-wz)$ and

$$f(w,z) = \frac{(w-1)^{A+B}(\zeta-w)^{A-B}(1-z)^{A-B}(z-\zeta^{-1})^{A+2B}}{(1-wz)^A}.$$

From (4.2) we get $I_n = (-k)^{An}(k+1)^{(A+2B)n}\tilde{I}_n$. Clearly $f(w,z), g(w,z)$ are analytic on $\Delta = \{(w,z)\in \mathbf{C}^2; wz\neq 1\}$ and the integral $\tilde{I}_n$ over $\alpha\times\beta$ converges absolutely by Lemma 4.1 for any rectifiable paths α,β lying in the upper-half plane satisfying the boundary conditions stated in Lemma 4.2. Put $\mu = B/A \in (0,1)$ for brevity. We will show successively that these functions satisfy the hypotheses stated in Section 1 for any $k\in\mathbf{N}$ and any $\mu\in\big(0, 3-2\sqrt{2}\,\big)$.

On Hypothesis A. Let D be the region obtained by omitting the two half-lines $(-\infty,\lambda_-]$ and $[\lambda_+,\infty)$ from the complex plane $\mathbf{C}$ where $\lambda_\pm$ are the two real roots of $(1/z-1)(1/z-\zeta)+(\mu/2k)^2=0$ satisfying $\zeta^{-1}<\lambda_-<\lambda_+<1$. Put

$$w_\pm(z) = \frac{1}{z} + \frac{\mu}{2k} \pm \sqrt{\left(\frac{1}{z}-1\right)\left(\frac{1}{z}-\zeta\right)+\left(\frac{\mu}{2k}\right)^2}$$

respectively, being the two roots of

$$\frac{1+\mu}{w-1} + \frac{1-\mu}{w-\zeta} = \frac{z}{wz-1}, \tag{6.1}$$

where the square root $\sqrt{u}$ is uniquely determined by $\sqrt{r}e^{i\theta/2}$ for any $u = re^{i\theta}$ with $r\geq 0$ and $0\leq\theta<2\pi$. For brevity, we denote by $H_\pm$ the upper-half and the lower-half planes and put

$$E_\pm = \left\{ z\in H_\pm; \left| z - 1 - \frac{1+\mu}{2k}\right| < \frac{\sqrt{1-\mu^2}}{2k}\right\}$$

respectively. Then it is easily verified that w_+ maps diffeomorphically D, H_+ onto H_+, E_+ respectively; similarly w_- maps diffeomorphically D, H_- onto H_-, E_- respectively. Since $1,\zeta\notin w_\pm(D)$ and $zw_\pm(z)\neq 1$ for any $z\in D$, it follows that (6.1) holds if and only if $\frac{\partial f}{\partial w}(w,z)=0$ for each $z\in D$; hence $\frac{\partial f}{\partial w}\big(w_\pm(z),z\big)\equiv 0$ on D. The hypothesis A is thus fulfilled by each $w_\pm(z)$. Note that $w_+(\bar{z}) = \overline{w_-(z)}$ and

$$\left(w_+(z) - 1 - \frac{1+\mu}{2k}\right)\left(w_-(z) - 1 - \frac{1+\mu}{2k}\right) = \frac{1-\mu^2}{4k^2}$$

for any $z\in D$.

On Hypothesis B. We show that the analytic function $F(z)\equiv f(w_+(z),z)$ on D possesses a unique saddle in D. Since $z\in D$ is a saddle of $F(z)$ if and only if

$$\frac{1-\mu}{z-1} + \frac{1+2\mu}{z-\zeta^{-1}} = \frac{w_+(z)}{zw_+(z)-1}, \tag{6.2}$$

it follows from (6.1) and (6.2) that any saddle z of $F(z)$ must satisfy the following cubic equation with real coefficients:

$$T(z) \equiv T_0(z) + \mu T_1(z) + \mu^2 T_2(z) = 0 \tag{6.3}$$

where

$$\begin{aligned} T_0(z) &= \zeta^2 z^3 - 3\zeta z + \zeta + 1, \\ T_1(z) &= (2\zeta z - 3\zeta + 1)(\zeta z^2 - 1), \\ T_2(z) &= \zeta z(z-1)(\zeta z - 2\zeta + 1). \end{aligned}$$

Then obviously $T(x) = 0$ has a unique negative root since $T(0) > 0$ and $T'(x) = 0$ has a positive root at which $T(x)$ attains its local minimum. It is easily seen that $T(x) > 0$ on $[0, \sqrt{1/\zeta}\,] \cup [2 - 1/\zeta, \infty)$. Now, for any $x \in (\sqrt{1/\zeta}, 1]$, we have

$$T(x) \geq T_0(x) + \mu T_1(x) > T_0\left(\sqrt{\frac{1}{\zeta}}\right) + \mu T_1(1) = (\sqrt{\zeta} - 1)^2 - \mu(\zeta - 1)^2 > 0,$$

since $\mu < 3 - 2\sqrt{2}$ and $T_2(x) \geq 0$ on this interval. Moreover, if $x \in (1, 2 - 1/\zeta)$, then

$$|T_2(x)| \leq \int_1^x |T_2'(t)|\ dt < \zeta \int_1^x (2\zeta - 1 - \zeta t)(2t - 1)\ dt < \frac{k+3}{k^2(k+1)} \leq \frac{2}{k^2};$$

hence $T(x) \geq T_0(1) + \mu T_1(1) - 2\mu^2/k^2 = (1 - \mu - 2\mu^2)/k^2 > 0$. Therefore (6.3) has two conjugate roots $z_0, \overline{z_0}$ with $\mathrm{Im}(z_0) > 0$ and there are no other roots in D. From (6.2) and (6.3) we get

$$w_+(z) = \frac{(2+\mu)\zeta z - \mu(2\zeta - 1) - \zeta - 1}{(1+\mu)\zeta z^2 - \mu(2\zeta - 1)z - 1} = \zeta\big((1+\mu)z - \mu\big); \tag{6.4}$$

this implies that $\overline{z_0}$ does not satisfy (6.2) since $w_+(D) = H_+$. Hence z_0 is a unique saddle of $F(z)$ in D, as required.

We next show the non-vanishing of $\frac{\partial^2 f}{\partial w^2}(w_+(z), z)$ for any $z \in D$. Suppose, on the contrary, that $\frac{\partial^2 f}{\partial w^2}(w_1, z_1) = 0$ for some $z_1 \in D$ where $w_1 = w_+(z_1) \in H_+$. Since

$$\frac{\frac{\partial^2 f}{\partial w^2}(w,z)}{f(w,z)} - \left(\frac{\frac{\partial f}{\partial w}(w,z)}{f(w,z)}\right)^2 = -\frac{A+B}{(w-1)^2} - \frac{A-B}{(w-\zeta)^2} + \frac{Az^2}{(wz-1)^2}$$

for any $(w, z) \in \Delta$, we have

$$\frac{1+\mu}{(w_1-1)^2} + \frac{1-\mu}{(w_1-\zeta)^2} = \left(\frac{z_1}{w_1 z_1 - 1}\right)^2 = \left(\frac{1+\mu}{w_1 - 1} + \frac{1-\mu}{w_1 - \zeta}\right)^2$$

and we get the quadratic equation

$$\mu(1-\mu)(w_1-1)^2 - 2(1-\mu^2)(w_1-1)(w_1-\zeta) - \mu(1+\mu)(w_1-\zeta)^2 = 0,$$

which implies that $w_1 \in \mathbf{R}$, which is a contradiction. Similarly we can show that $\frac{\partial^2 f}{\partial w^2}(w_-(z), z) \neq 0$ for any $z \in D$.

To show the non-vanishing of $\mathrm{Hess}_f(w_0, z_0)$ we consider $G(w) \equiv F(w_+^{-1}(w))$, being analytic in H_+. In fact $G(w)$ is a rational function whose zero points and poles are all real. To see this, using the relation

$$w_+^{-1}(w) = \frac{2w - \zeta - 1 - \dfrac{\mu}{k}}{w^2 - \dfrac{\mu}{k}w - \zeta}$$

coming from (6.1), we get

$$G(w) = \zeta^{-A-2B}\ \frac{(w-1)^A(\zeta - w)^{A+B}\left(w - 1 - \dfrac{\mu}{k}\right)^{A-B}\left(\zeta + \dfrac{\mu}{k} - w\right)^{A+2B}}{\left(w^2 - \dfrac{\mu}{k}w - \zeta\right)^{A+B}}.$$

Since w_+ maps the valley set $V(F; z_0) \subset D$ diffeomorphically onto $V(G; w_0) \cap H_+$, it follows that w_0 is a unique saddle of $G(w)$ in H_+ and that $\mathrm{ord}(G; w_0) = \mathrm{ord}(F; z_0)$. On the other hand, it is easily seen that $G(w)$ has at least two real saddles $w_+(\lambda_+) \in (1, 1 + \mu/k)$ and $w_+(\lambda_-) \in (\zeta, \zeta + \mu/k)$. By noticing that w is a saddle of $G(w)$ if and only if w satisfies a certain algebraic equation with real coefficients of degree five, we have

$$2 + 2(\mathrm{ord}(G; w_0) - 1) \le 5;$$

hence $\mathrm{ord}(F; z_0) = \mathrm{ord}(G; w_0) = 2$. This means that $\mathrm{Hess}_f(w_0, z_0) \neq 0$ and Hypothesis B is thus fulfilled.

On Hypothesis C. We need the following lemma concerning the behavior of $|F(x)|$ on the segment (λ_-, λ_+).

Lemma 6.1. *Put $a(x, y) = |F(x + iy)|$ for $x \in (\lambda_-, \lambda_+)$ and $y \in \mathbf{R}$. Then there exists a $\lambda^* \in (\lambda_-, \lambda_+)$ such that $a(x, 0)$ is monotone increasing on $(\lambda_-, \lambda^*]$ and monotone decreasing on $[\lambda^*, \lambda_+)$. Furthermore $\frac{\partial a}{\partial y}(x, 0) < 0$ for any $x \in (\lambda_-, \lambda_+)$.*

Proof. By the definition of $w_+(x)$ we get

$$w_+(x) = \frac{1}{x} + \frac{\mu}{2k} + i\sqrt{\left(\frac{1}{x} - 1\right)\left(\zeta - \frac{1}{x}\right) - \left(\frac{\mu}{2k}\right)^2}$$

for $x \in (\lambda_-, \lambda_+)$, from which it follows that

$$|F(x)| = \frac{(1+\mu)^{(A+B)/2}(1-\mu)^{(A-B)/2}}{k^A\,\zeta^{B/2}} \cdot \frac{(1-x)^{A-B/2}(x - \zeta^{-1})^{A+3B/2}}{x^A}. \tag{6.5}$$

As x varies in $(\zeta^{-1}, 1)$ the right-hand side of (6.5) attains its maximum at λ^*, which is a unique positive zero point of $Y(t) = 2(1+\mu)\zeta t^2 - \mu(3\zeta - 1)t - 2$. Then it can be seen that $Y(\lambda_-) < 0 < Y(\lambda_+)$ since $\mu \in (0, 3 - 2\sqrt{2}\,)$; hence $\lambda^* \in (\lambda_-, \lambda_+)$.

Moreover we have

$$\begin{aligned}\frac{\partial a}{\partial y}(x, 0) &= -|F(x)| \cdot \mathrm{Im}\left(\frac{F'(x)}{F(x)}\right) = A|F(x)| \cdot \mathrm{Im}\left(\frac{w_+(x)}{x w_+(x) - 1}\right)\\ &= -A|F(x)| \cdot \frac{\sqrt{\left(\dfrac{1}{x} - 1\right)\left(\zeta - \dfrac{1}{x}\right) - \left(\dfrac{\mu}{2k}\right)^2}}{(1-x)(\zeta x - 1)} < 0.\end{aligned}$$

This completes the proof. ∎

By Remark 2 stated in the end of the previous section, the valley set $V(F; z_0)$ consists of two connected components, say K^0 and K^1, and $V(G; w_0)$ consists of two connected components W^0 and W^1 satisfying $w_+(K^j) = W^j \cap H_+$ for $j = 0, 1$. Note that each $K^j \subset D$ contains no zero points of $F(z)$. Since W^0 and W^1 are symmetric with respect to the real axis, they enclose some connected component of $U(G; w_0)$. This means that 1 and $1 + \mu/k$ cannot belong to different W^j's and the same thing holds for ζ and $\zeta + \mu/k$, since there are no poles between such zero points. So we assume that W^0 contains either 1 or $1 + \mu/k$ and that W^1 contains either ζ or $\zeta + \mu/k$; anyway $\zeta^{-1} \in \overline{K^0}$ and $1 \in \overline{K^1}$.

We first show that K^0_+ is connected. For otherwise, it follows from Lemma 6.1 that there exists some region U in H_- such that $(\partial U)_- \subset \partial K^0$ and $(\partial U)_{\mathbf{R}} \subset (\lambda_-, \lambda_+)$. Thus $|F(z)|$ attains its maximum at a certain point in U since $\frac{\partial a}{\partial y}(x, 0) < 0$ on $(\partial U)_{\mathbf{R}}$, which contradicts the maximum principle. We can show similarly the connectedness of K^1_+.

We next show that $\zeta^{-1} \in \overline{K^0_+}$. For otherwise, we have $\zeta^{-1} \in \overline{K^0_-}$ and $K_{\mathbf{R}} \subset (\lambda_-, \lambda_+)$. Then it follows from Lemma 6.1 that $(\lambda_-, \lambda_- + \varepsilon) \subset K_{\mathbf{R}}$ for some $\varepsilon > 0$; in particular

$$|G(w_+(\lambda_-))| = |F(\lambda_-)| < |F(z_0)| = |G(w_0)|.$$

This implies that both 1 and $1 + \mu/k$ are contained in W^0 since $|G(x)| < |G(w_0)|$ for any $x \in [1, 1 + \mu/k]$; hence $\zeta^{-1} \in \overline{K^0_+}$, a contradiction. We can show similarly that $1 \in \overline{K^1_+}$.

The above results mean that one can find a rectifiable path $\gamma : (-1, 1) \to H_+$ satisfying the conditions in Hypothesis C and the boundary conditions for β in Lemma 4.2.

On Hypotheses D, E and F. Hypothesis D is clearly fulfilled since

$$f_z(w) \equiv f(w, z) = c_z \, \frac{(w-1)^{A+B}(\zeta - w)^{A-B}}{\left(w - \dfrac{1}{z}\right)^{A}}$$

where c_z is a non-zero complex constant independent of w, for an arbitrarily fixed $z \in D$. $f_z(w)$ is analytic in $D_z = \{w \in \mathbf{C}; w \neq 1/z\}$ and satisfies the conditions stated in Lemma 5.3 for any $z \in H_+$. Note that $f_z(w)$ has exactly two saddles $w_\pm(z)$ in $H_\pm$ respectively, which are always normal since $\frac{\partial^2 f}{\partial w^2}(w_\pm(z), z) \neq 0$.

By Lemmas 5.2 and 5.6 the valley set $V(f_z; w_+(z))$ consists of two connected components, say K^0_z and K^1_z, such that $1 \in K^0_z$ and $\zeta \in K^1_z$. Since each $\left(K^j_z\right)_+$ is connected by Lemma 5.5 one can find a rectifiable path $\delta_z : (-1, 1) \to H_+$ for each $z \in \gamma \subset H_+$ satisfying the conditions in Hypothesis E and the boundary conditions for α in Lemma 4.2. Moreover the lengths of δ_z can be made uniformly bounded for $z \in \gamma$. Then Hypothesis F is clearly fulfilled with $\ell = 1$ by Lemma 4.1.

We thus conclude that

$$\lim_{n\to\infty} \frac{1}{n} \log |I_n| = A \log k + (A + 2B) \log(k + 1) + \log |f(w_0, z_0)|. \tag{6.6}$$

Remark 3. As z varies in D, the valley set $V(f_z; w_+(z))$ changes continuously in the sense of the Hausdorff metric. To see this, suppose that z^* is a discontinuity point of $V(f_z; w_+(z))$ when z varies along a certain continuous curve lying in D. Since $|f_z(w)|$ attains neither a local maximum nor a positive local minimum, it follows that another saddle $w_-(z^*)$ must be contained in the boundary of some connected component K of $V(f_{z^*}; w_+(z^*))$ and that $V(f_{z^*}; w_-(z^*))$ must have a connected component K' with $K \cap K' = \varnothing$. This is a contradiction since $n(f_{z^*}; w_-(z^*)) = 1$ by Lemma 5.6. On the other hand, one can observe discontinuity phenomena of the valley set $V(f_z; w_-(z))$. For example, in the case in which $k = 1$ and $\mu = 1/10$, $V(f_z; w_-(z)) \supset V(f_z; w_+(z))$ for $z = 1 + ti$ with $t > t^*$, while $V(f_z; w_-(z))$ is contained in one connected component of $V(f_z; w_+(z))$ for $0 < t \le t^*$ where t^* is approximately 0.0013344216.

The asymptotic behavior of the common coefficient a_n in (4.6) can also be obtained by applying the $\mathbf{C}^2$-saddle method to the same functions $f(w,z)$ and $g(w,z)$. However this is much easier than the previous case. Indeed it follows from (2.6) that, for any positive numbers r and R with $rR > 1$,

$$\begin{aligned} a_n &= -\frac{1}{2\pi i}\int_C P_n(u)\, Q_n\left(\frac{1}{u}\right)\frac{du}{u} \\ &= \frac{k^{2(A+B)n}}{4\pi^2}\iint_{C\times C'}\left(\frac{(u-1)^{A-B}(\zeta-u)^{A+2B}(w-1)^{A+B}(\zeta-w)^{A-B}}{u^{A+B}(w-u)^A}\right)^n\frac{du\,dw}{uw} \end{aligned}$$

where C and C' are the circles centered at the origin with radii $1/R$ and r respectively. Substituting $z = 1/u$, we get

$$a_n = -\frac{(-k)^{An}(k+1)^{(A+2B)n}}{4\pi^2}\iint_{C'\times C''} g(w,z)\big(f(w,z)\big)^n\,dw\,dz$$

where C'' is the circle centered at the origin with radius R. Then it is easily seen that $|f(w,z)| \le |f(-r,-R)|$ for any $(w,z) \in C' \times C''$ and the equality holds if and only if $(w,z) = (-r,-R)$. Moreover it can be seen that, as r and R vary with $rR > 1$, $|f(-r,-R)|$ attains its minimum at $(r,R) = (r_0,R_0)$ satisfying $T(-R_0) = 0$ and $r_0 = \zeta((1+\mu)R_0 + \mu)$. This means that C' and C'' play the same roles as δ_z and γ in Hypotheses E and C respectively. The situation is simpler than the previous case since C' is independent of $z \in C''$. Therefore the $\mathbf{C}^2$-saddle method can be applied to the functions $f(w,z)$ and $g(w,z)$ at $(-r_0,-R_0)$; hence

$$\lim_{n\to\infty}\frac{1}{n}\log|a_n| = A\log k + (A+2B)\log(k+1) + \log|f(-r_0,-R_0)|. \tag{6.7}$$

section*7. Main result

The simultaneous rational approximation (4.6) can be written as

$$\begin{cases} q_n \log\zeta - p_n = \varepsilon_n, \\ q_n \log^2\zeta - r_n = \delta_n, \end{cases}$$

with $p_n = -b_n M_n, q_n = a_n M_n, r_n = 2c_n M_n$,

$$\varepsilon_n = -\frac{M_n}{\pi}\operatorname{Im}(I_n) \quad \text{and} \quad \delta_n = -2M_n\left(\operatorname{Re}(I_n) - \frac{\log\zeta}{\pi}\operatorname{Im}(I_n)\right)$$

where $M_n = (D_{(A+B)n}/\Lambda_n^{**})D_{(A+B)n}\Delta_{(A+B)n}$. Combining (4.7), (6.6) and (6.7) it follows that

$$\lim_{n\to\infty}\frac{1}{n}\log|q_n| = \sigma(k,\mu)$$

TABLE 1.

logarithm at	k	μ^{-1}	non-quadraticity measure
2	1	10	25.0463
3/2	2	13	10.9199
4/3	3	16	8.5943
5/4	4	19	7.5869
6/5	5	22	7.0087
7/6	6	25	6.6269
8/7	7	28	6.3527
9/8	8	31	6.1444
10/9	9	34	5.9795
11/10	10	37	5.8451

and

$$\max\left\{\limsup_{n\to\infty}\frac{1}{n}\log|\varepsilon_n|, \quad \limsup_{n\to\infty}\frac{1}{n}\log|\delta_n|\right\} \le -\tau(k,\mu)$$

where

$$\sigma(k,\mu) = 2(A+B) - \chi(A,B) + A\log k + (A+2B)\log(k+1) + \log|f(-r_0,-R_0)|$$

and

$$\tau(k,\mu) = -2(A+B) + \chi(A,B) - A\log k - (A+2B)\log(k+1) - \log|f(w_0,z_0)|.$$

Then by Lemma 2.3 we have

Theorem 7.1. *For any $k\in\mathbf{N}$ and $\mu\in(0,3-2\sqrt{2})$ satisfying $\tau(k,\mu)>0$, the number $\log(1+1/k)$ has a non-quadraticity measure $\sigma(k,\mu)/\tau(k,\mu)+1$; more precisely, for any $\varepsilon>0$, there exists an effective positive constant $H_0(\varepsilon)$ such that*

$$\left|\log\left(1+\frac{1}{k}\right)-\xi\right| \ge H^{-\sigma(k,\mu)/\tau(k,\mu)-1-\varepsilon}$$

for any quadratic number ξ with $H\equiv H(\xi)\ge H_0(\varepsilon)$.

For example, in the case in which $k=1$ and $\mu=1/10$, a numerical computation shows that the cubic equation

$$\frac{1}{2}T(z) = 242z^3 - 55z^2 - 317z + 175 = 0$$

has a negative root $-R_0$ and complex roots $z_0, \overline{z_0}$ with $R_0 = 1.2630812985...$ and $z_0 = 0.7451770129... + 0.1312713306...i$. Using the relations $r_0 = (11R_0+1)/5$ and $w_0 = (11z_0-1)/5$ we then get

$$12\log 2 + \log|f(-r_0,-R_0)| < 41.94932, \qquad 12\log 2 + \log|f(w_0,z_0)| < -20.48239.$$

Since $\chi(10) > 4.010263$ by (4.8), it follows from Theorem 7.1 that $\log 2$ has a non-quadraticity measure

$$\frac{\sigma(1,1/10)}{\tau(1,1/10)} + 1 < \frac{20.48239+41.94932}{-22+4.010263+20.48239} < 25.0463.$$

Numerical examples for $1\le k\le 10$ are listed in Table 1.

If $1 \le k \le 13$, then the minimum of the non-quadraticity measure $\sigma(k,\mu)/\tau(k,\mu) + 1$, as μ varies, seems to be attained at $\mu^{-1} = 3k+7$.

Remark 4. Since $a_n \in \mathbf{Z}$ and $b_n \in \mathbf{Z}/(D_{(A+B)n}/\Lambda_n^{**})$, it follows from the first equality in (4.6) that

$$a_n \tilde{M}_n \ \log\zeta + b_n \tilde{M}_n = -\frac{\tilde{M}_n}{\pi}\,\mathrm{Im}(I_n)$$

where $\tilde{M}_n = D_{(A+B)n}/\Lambda_n^{**}$, giving a single rational approximation to $\log\zeta$. However this will not give a good irrationality measure of $\log\zeta$. For example, in the case $k=1$, this gives an irrationality measure 4.53406 for $\log 2$ at $\mu = 1/7$, worse than the known best measure 3.8913997... (see [11] and [4, Theorem 4.1]).

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